Hadamard Walsh and Paley Ordered DFWHT: A Study and Implementation on FPGA

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Abstract— Walsh transform shows an ascending of sequency analogous to Fourier transform unlike random sequency of Hadamard transform. This provides the benefit of Walsh transform computation using fast algorithm which is identical to the FFT (Fast Fourier Transform) leading to the efficient hardware realization. The kernel of Walsh transformation being symmetric matrix with orthogonal rows and columns, the same algorithm can be used for both the 2-D forward and inverse Walsh transforms without modification. Hence, only one hardware block is sufficient to implement both forward and inverse transform which is not possible in DCT (Discrete Cosine Transform) based algorithm. Moreover, the wide usage of Walsh codes for implementation of CDMA (Code Division Multiple Access) in wireless communication makes Walsh transform more attractive. This paper discusses the introduction and basic overview of the Walsh-Hadamard transform, its properties, ordering, applications and advantages. When the Walsh and Hadamard matrices need to be used in digital design, we can store them in a ROM and call up the values of the matrices whenever and wherever required. Finally the implementation of the different ordering of the WHT is simulated using Xilinx FPGA using ISE 14.6. Each Hadamard ordered matrix, Walsh ordered matrix and Dyadic ordered matrix in the ROM uses 4 slices only for their storage.

Index Terms— Discrete Fast Walsh-Hadamard Transform, Digital signal processing, FPGA

I. INTRODUCTION

TN the present day for digital signal processing there are I many transforms available as those of the Discrete Fourier Transform (DFT), Fast Fourier Transform (FFT), Discrete Cosine Transform (DCT), Discrete Fast Walsh Hadamard Transform (DFWHT), and many more.First and foremost, a Fourier transform of a signal tells us what frequencies are present in our signal and in what proportions. The Walsh-Hadamard transform (WHT) is a suboptimal, non-sinusoidal, orthogonal transformation that decomposes a signal into a set of orthogonal, rectangular waveforms called Walsh functions. The transformation has no multipliers and is real because the amplitude of Walsh (or Hadamard) functions has only two values, +1 or -1.WHTs are used in many different applications, such as power spectrum analysis, filtering, processing speech and medical signals, multiplexing and coding in communications, characterizing non-linear signals,

solving non-linear differential equations, and logical design and analysis.In a Walsh-Hadamard matrix the kernels are made up of only "+1s" and "-1s". Thus the Walsh-Hadamard Transform (WHT) can be implemented by virtually ruling out the time consuming and tiresome multiplication operation as, even if a need of multiplication arises the numbers being multiplied with the matrices will be the number itself or the number itself with a negative sign before as the kernels of the Hadamard and Walsh matrices are only "+1s" and "-1s".

The Hadamard matrix is named after the French mathematician Jacques Hadamard. The matrix elements comprises of only "+1s" and "-1s" and is a square matrix and has equal number of "+1s" and "-1s" except for the first row and column of the matrix. In a Hadamard matrix the rows are mutually orthogonal. Sequency of a row or a column of the kernel is given by the decimal representation of the Gray code of the bit reversed binary values of the corresponding row or the column index. The Walsh matrix was proposed by Joseph L. Walsh in the year 1923. The Walsh matrix is similar to that of the Hadamard matrix. The Walsh matrix also has orthogonal rows and the elements of the matrix comprises of "+1s" and "-1s". The Walsh matrix is again necessarily a square matrix and the "+1s" and "-1s" are arranged into orthogonal rows. Similar to the Hadamard matrix in the Walsh matrix too the "+1s" and "-1s" are equal in each row and column except for the first row and column of the matrix.

II. PROPERTIES OF WALSH HADAMARD MATRICES

The Walsh and Hadamard functions form a complete orthogonal set; hence they can be used to represent signals in much the same way as the sinusoidal functions are used in analog signal processing [10]. Walsh functions are two valued. If the "+1s" are represented as "0s"and the "-1s" are represented as "1s" then they can be incorporated into the digital designs of the systems. This matches the binary nature of logic circuitry and digital computer very well, thus indicating a potentially great advantage over the sinusoidal functions when implemented using logic circuits or digital computers are intended [10].

In a Discrete Walsh Transform, if M is a real valued square matrix and I is the identity matrix then:

- 1. The transposed matrix can be found by interchanging rows and columns.
- 2. M is orthogonal if $MM^T = M^TM = I$
- 3. M is orthogonal up to the constant k if $MM^{T} = M^{T}M = kI$
- 4. M^{-1} is the inverse of M if $MM^{-1} = M^{-1}M = I$
- 5. M has its inverse if the column vectors of M are independent. The Walsh Hadamard Transform is given by the formula:

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$$Y = H_N X \tag{1}$$

where, Y is the transformed matrix, H represents the Hadamard matrix, subscript N represent the order of the Hadamard matrix, and X represent data matrix. An example for the same is shown below:

And Y is the transformed matrix after the transformation with H_8 is shown below:

where, j represents the column number.

III. KRONECKER PRODUCT

The Kronecker product is named after the German mathematician Leopold Kronecker and is represented by \otimes . This is used for the multiplication of two matrices irrespective of size of the composing matrices. The Kronecker product is totally different from the simple matrix multiplication. If A is a matrix of size (m,n) and B is a matrix of size (p,q) then the Kronecker product of the matrices A and B is given by a matrix whose size is (mp,nq).

$$A_{mn} \otimes B_{pq} = \begin{bmatrix} a_{11}b_{11} & \cdots & a_{1n}b_{1q} \\ \vdots & \ddots & \vdots \\ a_{m1}b_{p1} & \cdots & a_{mn}b_{pq} \end{bmatrix}$$
(3)

On expanding the above matrix:

$$\mathbf{A}_{mn}\mathbf{B}_{pq} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{1n}b_{1q} & \cdots & a_{1n}b_{11} & a_{1n}b_{12} & \cdots & a_{1n}b_{1q} \\ a_{11}b_{21} & a_{11}b_{22} & \cdots & a_{1n}b_{2q} & \cdots & a_{1n}b_{21} & a_{1n}b_{22} & \cdots & a_{1n}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ a_{11}b_{p1} & a_{11}b_{p2} & \cdots & a_{1n}b_{pq} & \cdots & a_{1n}b_{p1} & a_{1n}b_{p2} & \cdots & a_{1n}b_{pq} \\ \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots & & \vdots \\ a_{1n}b_{11} & a_{m1}b_{12} & \cdots & a_{m1}b_{1q} & \cdots & a_{mn}b_{11} & a_{mn}b_{12} & \cdots & a_{mn}b_{1q} \\ a_{m1}b_{21} & a_{m1}b_{22} & \cdots & a_{m1}b_{2q} & \cdots & a_{mn}b_{21} & a_{mn}b_{22} & \cdots & a_{mn}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{n1} & a_{m1}b_{22} & \cdots & a_{m1}b_{2q} & \cdots & a_{mn}b_{21} & a_{mn}b_{22} & \cdots & a_{mn}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{n1} & a_{m1}b_{p2} & \cdots & a_{m1}b_{pq} & \cdots & a_{mn}b_{p1} & a_{mn}b_{p2} & \cdots & a_{mn}b_{pq} \end{bmatrix}$$

Where, $a_{mn}b_{pq}$ means simple multiplication of a_{mn} and b_{pq} elements. The rule followed in Kronecker product is that each element of the matrix is multiplied with all the elements of the other matrix.

IV. HADAMARD ORDERED, WALSH ORDERED AND DYADIC ORDERED SIGN CHANGES IN EACH ROW OF MATRIX FOR N=8

The sign changes in each row for Hadamard Order, Walsh Order and Dyadic Ordered matrices for N=8 is shown in Table I below:

TABLE I Hadamard ordered, Walsh ordered and Dyadic ordered sign changes in each row of matrix for N=8								
HadamardHadamardHadamardOrderedOrderedOrdered(Natural Order)(Natural Order)Order)								
0	0	0						
7	7	7						
3	3	3						
4	4	4						
1	1	1						
6	6	6						
2	2	2						
5	5	5						

V. WALSH HADAMARD MATRIX GENERATION

The Walsh Hadamard Matrix can be generated by using a recursive method. The formula for the same is shown below:

$$H_{N} = (H_{2} \otimes I_{N/2}) (I_{2} \otimes H_{N/2})$$
(5)

where, H_N represents the Hadamard matrix and N represent the order of the Hadamard matrix and can be any value such that $N \ge 1$. I represents the index matrix. The recursive method generates a Hadamard matrix of the simplest order. An example of H_4 is shown below using (5):

$$H_4 = (H_2 \otimes I_{4/2}) (I_2 \otimes H_{4/2})$$
 (6)

$$H_4 = (H_2 \otimes I_2) (I_2 \otimes H_2)$$
(7)

VI. ORDER OF WALSH-HADAMARD MATRICES

The Walsh–Hadamard matrices are differentiated from each other according to the order of the matrices. The matrices can be classified into three major orders as:

A. Natural Ordered

The natural order is also known as the Hadamard order. This can be generated by Kronecker product.

$$H_{N} = \underbrace{H_{2} \otimes \dots \otimes H_{2}}_{n \text{ times}} H_{2}$$
(10)

where, N=2ⁿ and N is called the order of the matrix. An example of H₈ is shown below using (10), where N=8, $N = 2^3, n = 3$

$$H_8 = H_2 \otimes H_2 \otimes H_2 \tag{11}$$

$$=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
(12)

The numbers represented by \rightarrow beside each row of the matrix denotes the number of times there has been a sign change in that particular row of the matrix. The natural order can also be generated by sampling the Walsh function.

B. Sequency Ordered

The sequency order is also known as the Walsh order. Sequency is defined as one-half the average number of the zero crossings per unit time interval and is closely related to the number of sign changes in the rows of the Hadamard matrix. An example of H_8 Hadamard matrix in its sequency ordered form is shown below:

The numbers represented by \rightarrow beside each row of the matrix denotes the number of times there has been a sign change in that particular row of the matrix.

C. Dyadic Ordered

The dyadic order is also known as the Paley order. By premultiplying the naturally ordered Hadamard matrix with the bit-reversed order matrix, we obtain the dyadic ordered matrix. Extra computations involving the Gray code-to-binary decoding are required at the end, if a sequency-ordered spectrum is required. The dyadic order matrix can be generated using the Matlab code [11] [12]. An example of H₈ Hadamard matrix in its dyadic ordered form is shown below:

	1	1	1	1	1	1	1	1	$ \rightarrow 0$	
	1	1	1	1	- 1	- 1	- 1	- 1	\rightarrow 7	
	1	1	- 1	- 1	1	1	- 1	- 1	$\rightarrow 3$	
11	1	1	- 1	- 1	- 1	- 1	1	1	$\rightarrow 4$	(15)
п ₈ =	1	- 1	1	- 1	1	- 1	1	- 1	$\rightarrow 1$	(15)
	1	- 1	1	- 1	- 1	1	- 1	1	$\rightarrow 6$	
	1	- 1	- 1	1	1	- 1	- 1	1	$\rightarrow 2$	
	1	- 1	- 1	1	- 1	1	1	- 1	$\rightarrow 5$	

numbers represented by \rightarrow beside each row of the matrix denotes the number of times there has been a sign change in that particular row of the matrix.

For the digital applications of these orders of the Walsh-Hadamard matrices the same when done on Xilinx platform the output waveforms of a Natural order matrix is shown in Fig. 1, a Sequency order matrix is shown in Fig. 2, and a Dyadic ordered matrix is shown in Fig. 3.

VII. APPLICATIONS

The Walsh Hadamard Transform has been very useful in various fields for the simplicity if the calculations and the fast computational speeds. To list few of the applications:

- 1. Randomness of a finite sequence can be measured [2],
- 2. Testing of random number sequence [5],
- 3. Solving first order differential equations [4],
- 4. Used in designing of logic gates [3],
- 5. Solving differential and integral equations [1],
- 6. It is an important tool in speech processing and error correction [8],
- 7. Calculation of Discrete Fourier Transform for implementing adaptive filters and spectrum filter realization [9],
- 8. Quantum Information processing and quantum cryptography [13][14],
- 9. Implementation of CDMA (Code Division Multiple Access) in wireless communications[15],
- 10. Signal processing [6], and many more [7][8].
- 11. Digital Watermarking [16]

VIII. EXPERIMENTAL RESULTS AND CONCLUSION

This paper discusses the introduction and basic overview of the Walsh-Hadamard transform, its properties, ordering, applications and advantages. When the Walsh and Hadamard matrices need to be used in digital design, we can store them

TABLE II ROM storage result of the Natural, Sequential and Dyadic ordered matrices										
	Natu	ral Order	Seque	ntial Order	Dyac	lic Order				
	Used	Available	Used	Available	Used	Available				
Number of slice	4	12480	4	12480	4	12480				

in a ROM and call up the values of the matrices whenever and wherever required. Finally the implementation of the different ordering of the WHT is simulated using Xilinx ISE 14.6. Each Hadamard ordered matrix, Walsh ordered matrix and Dyadic ordered matrix in the ROM uses 4 slices only for their storage. Table II shows the ROM storage details of the natural, sequency and dyadic ordered matrices. The natural, sequency and dyadic matrix waveforms are shown in Fig. 1,2, and 3 respectively. In the figure, the shaded portions show a high value state representing 1 and the unshaded portions show a low value state representing a 0. In figure a 0 corresponds to "+1" of the matrix and the 1 in figure corresponds to "-1" of the matrix.

Name	Value	0 us	1us	2 us	3 us	4us	5 us	6 us	7 us	18 us
🔻 🃑 row[2:0]	000	UUU	000	001	010	011	100	101	110	111
1 [2]	0									
1 🔓 [1]	0									
1 📴 [0]	0									
🔻 🃲 data[7:0]	0000000	XXXXXXXXXX	00000000	01010101	00110011	01100110	00001111	01011010	00111100	01101001
1 [7]	0									
1 [6]	0									
1 [5]	0									
1 [4]	0									
Цар	0									
1 [2]	0									
1 🔓 📖	0									
1 [0]	0									
_										

Fig. 1: Waveforms showing Natural ordered or Hadamard ordered matrices' rows and corresponding value in ROM

							-	U		
Name	Value	0 us	1us	2 us	3 us	4 us	5 us	6 us	7 us	8 us
🔻 🃑 row[2:0]	000	000	000	001	010	011	100	101	110	111
1 [2]	0	1								
կ ան	0									
1 [0]	0									
🔻 🌃 data[7:0]	00000000	XXXXXXXXXX	00000000	00001111	00111100	00110011	01100110	01101001	01011010	01010101
ل 🔁 🔁	0									
1 [6]	0									
رها (۵)	0									
լե [4]	0									
1 <mark></mark> (3)	0									
1 [2]	0									
կ ալո	0									
1	0									

Fig. 2: Waveforms showing Sequency ordered or Walsh ordered matrices'rows and corresponding value in ROM

<u> </u>							-	e		
Name	Value	0 us	1us	2 us	3 us	4us	5 us	6 us	7 us	18 us
🔻 🃑 row[2:0]	000	UUU	000	001	010	011	100	101	110	111
1 [2]	0									
1 [1]	0						1			
1 📴 [0]	0									
🔻 🃲 data[7:0]	00000000	XXXXXXXXX	00000000	00001111	00110011	00111100	01010101	01011010	01100110	01101001
رم 🖫 🕞	0									
1 [6]	0									
1 🔓 (5)	0									
لو [4]	0									
1 [3]	0				1		1			
1 [2]	0									
1 🔓 (1)	0									
1 m (0)	0									
	-									

Fig. 3: Waveforms showing Dyadic ordered or Paley ordered matrices'rows and corresponding value in ROM

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